Critical percolation and lack of self-averaging in disordered models

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Abstract - Lack of self-averaging originates in many disordered models from a fragmentation of the phase space where the sizes of the fragments remain sample-dependent in the thermodynamic limit. On the basis of new results in percolation theory, we give here an argument in favour of the conjecture that critical two dimensional percolation on the square lattice lacks of self-averaging.

The purpose of this note is to discuss a possible relationship between the arising of non self-averaging effects in disordered models and the occurrence of a percolative phase transition. More precisely we conjecture as follows: in two dimensional bond percolation on the square lattice non self-averaging effects are present at the critical point $p = p_c = \frac{1}{2}$, whereas above and below the critical probability the model is shown to be self-averaging. We also suggest that the phenomenon underlying the lack of self-averaging in many well studied models is a percolation-like phase transition: whenever non self-averaging effects are present these systems should be in a critical percolative state. A crucial role for the justification of these asserts will be played by the recent progress in two dimensional percolation theory achieved by Aizenman [1], Shchur and Kosyakov [2], and Cardy [3].

The appearance of non self-averaging effects in the low temperature phase has been the most interesting outcome of the replica approach to spin glasses [4]. It has later been recognized that such effects are present in a large class of even simpler models ranging from condensed matter theory to population biology [5], to dynamical systems' theory [6] and mathematics as well (see Ref. [7] for a unifying review of some of them). In Derrida's words, in

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the low temperature spin glasses "phase space can be thought of as if it was decomposed into infinitely many pure states α, the weights of which remain sample dependent even in the thermodynamic limit" [7]. This results in non vanishing sample-to-sample fluctuations of the weights W_α even in the thermodynamic limit, so that any no matter how large sample is never a good representative of the whole ensemble. Furthermore, one may find finite-weighted pure states in each sample, something which sounds strange since the condition $\sum_\alpha W_\alpha = 1$ must always be satisfied. The same holds for many other models. In some cases, the expression obtained for the fluctuations of the weights coincides with that obtained for spin glasses in particular limits (as pointed out in Ref. [7]), so that, again citing Derrida, it looks like "the spin glass problem, at least in its mean field version, belongs to a larger class of problems, and it would be interesting to develop a more general theory" [7] to treat them.

By now, the standard method to detect non self-averaging effects is that of studying the quantity

$$Y = \sum_\alpha W_\alpha^2. \tag{1}$$

It is possible to show (see Ref. [7] for explicative applications) that if both the ensemble average $\langle Y \rangle$, and the variance $\mathrm{var}(Y) = \langle Y^2 \rangle - \langle Y \rangle^2$ of Y are non zero in the thermodynamic limit, so that the probability density $\Pi(Y)$ remains "broad" when the system's size goes to infinity, then Y and consequently the weights W_α are non self-averaging. $\langle Y \rangle$ is the average of Y over all possible samples, that is over all possible realizations of disorder, represented by a number of quenched random variables.

Broken objects are perhaps the most intuitive and simplest models showing the same non self-averaging behaviour as spin glasses. Consider fragmenting a given object of size 1 into infinite pieces of sizes W_α according to a given breaking process. A sample corresponds to a specific rupture, hence to particular values of a set of quenched random variables on which the process depends. For some processes one finds that the sizes of the pieces lack of self-averaging, that is they remain sample dependent despite of the fact that the number of pieces is infinite and that $\sum_\alpha W_\alpha = 1$. This is the case of Derrida and Flyvbjerg's randomly broken object [8], where the breaking process depends on an infinite number of quenched random variables. We have recently shown that such a complex procedure is not necessary for non self-averaging effects to appear. In a geometrically broken object [9], where the breaking process depends on just one random variable p and the sizes of the pieces form a geometric sequence, the situation is seemingly less complicated than that of the randomly broken object, and yet the same non

self-averaging effects are present.

Let us now turn to bond percolation and consider a subset of the two dimensional square lattice $\mathcal{L}_2 = \mathbb{Z}^2$ given by a square $\mathcal{S}(N)$ of N^2 points, having N points on each side. Once the bonds have been assigned with the usual procedure, namely there is a bond between neighbouring lattice sites with probability p, the square is broken into clusters whose size is easily identifiable as the number of points forming them, and an isolated point is regarded as a cluster of size 1. The sizes normalized with the total number of points N^2 are the weights W_α, where α is an index running from 1 to the number of clusters formed. Clearly for each way of assigning the bonds and of forming the clusters one has $\sum_\alpha W_\alpha = 1$. From this viewpoint, percolation theory on $\mathcal{S}(N)$ studies an object (the square) broken into pieces (the clusters) according to the well known rule. The thermodynamic limit is immediately given by the $N \to \infty$ limit, in which $\mathcal{S}(N) \to \mathcal{L}_2$. In this limit percolation theory on the two-dimensional lattice, providing us with bounds and estimates for the probability that a piece is of a certain size in a sample produced with acertain value of p, is recovered.

A first problem concerns the ensemble in which averages should be evaluated. We could surely introduce disorder by treating p as a random variable, namely producing percolation lattice samples with a value of p chosen from a probability density $\rho(p)$ on the $[0, 1]$ interval, and calculate averages over all possible choices of p, as is done in the geometrically broken object, where the breaking process depends on just one variable as well. But since in bond percolation to a given value of p correspond infinitely many different breakings, one can define the ensemble of all possible ways of forming clusters with a given value of p and calculate averages within that ensemble. This "fixed p" ensemble average will be denoted by $\langle \dots \rangle_p$.

Getting back to bond percolation, let us consider the subcritical phase (for whatever concerns percolation theory in this paper we refer the reader to Ref. [10]; all following citations of basic results in subcritical and supercritical percolation are taken from that book), where no infinite-sized clusters are present almost surely and the average cluster size is thus finite (Chapter 5). It is intuitive that in the thermodynamic limit the probability that two randomly chosen lattice points belong to same cluster, that is Y, goes to zero, so that the clusters' weights are self-averaging. Nevertheless the proof that self-averaging holds is quite straightforward. If we denote by C_N the number of clusters (including isolated points) that are formed in the square $\mathcal{S}(N)$ and by $\langle n \rangle_p$ the average number of points in a cluster as a function of

p, it is possible to show[1] that whenever $p < p_c$

$$\langle Y \rangle_p \simeq \lim_{N\to\infty} \frac{1}{N^2} \frac{C_N}{N^2} \langle n^2 \rangle_p. \tag{2}$$

From the facts that in the $N \to \infty$ limit the ratio C_N/N^2 tends to an analytic function $\kappa(p)$ known as the number of open clusters per vertex (Chapter 4), and that $\langle n^2 \rangle_p$ is finite in the subcritical phase (Lemma 5.89 and in particular its consequences), follows that $\langle Y \rangle_p \to 0$ for all $p < p_c$ as $N \to \infty$.

For what concerns the supercritical phase, where there is one only infinite cluster almost surely and the average cluster size diverges (Chapter 6), an analogous argument yields the conclusion that, if we restrict ourselves to finite clusters,

$$\langle Y \rangle_p^{(\text{finite})} \to 0 \tag{3}$$

when $N \to \infty$ and $p > p_c$, which means that finite sized clusters do not contribute to the probability that two randomly chosen lattice points belong to the same cluster. For the unique infinite cluster we have simply that $\langle Y \rangle_p^{(\text{infinite})} \geq P_p(\infty)^2$, where $P_p(\infty)$ is the probability that a cluster has infinite size, which for $p > p_c$ is finite and positive.

Let's come to the critical point $p = p_c$. Up to recent times it was commonly believed that there exists exactly one infinite cluster in two dimensional square lattices at the critical point. This view has been changed by Aizenman, whose work [1] presents the following main results. The starting point is the concept of "spanning cluster", under which one understands a cluster that spans a large though finite region of the square lattice. For example, given the square $\mathcal{S}(N)$, a spanning cluster is a cluster that transverses it from left to right. Aizenman proved that when $p = p_c$:

1. the probability $\Sigma_N(\geq k)$ that $\mathcal{S}(N)$ is transversed (from left to right) by at least k distinct clusters is strictly positive for all finite k;

2. $\Sigma_N(\geq k)$ satisfies the inequalities

$$A \exp(-\alpha k^2) \leq \Sigma_N(\geq k) \leq \exp(-\beta k^2), \tag{4}$$

where A, α and β are different positive constants.

[1]The proof runs as follows: since the weight of a cluster of size n formed in the square $\mathcal{S}(N)$ is n/N^2, one writes $\langle Y \rangle_p = \lim_{N\to\infty} \sum_{n=1}^{N^2} (n/N^2)^2 f(n)$, where $f(n)$ is the average number of clusters of size n formed in the square. $f(n)$ may be approximated by $C_N P_p(n)$, $P_p(n)$ denoting the probability that a cluster contains n points. Hence one gets $\langle Y \rangle_p = \lim_{N\to\infty} \sum_n \frac{C_N}{N^4} n^2 P_p(n)$, from which formula (2) soon follows.

Furthermore he conjectured that the limit

$$\lim_{k\to\infty}\lim_{N\to\infty}\frac{1}{k^2}\log\Sigma_N(\geq k) \tag{5}$$

exists and is finite. Shchur and Kosyakov [2] have carried out numerical studies on the first limit, namely

$$\lim_{N\to\infty}\Sigma_N(\geq k)=\Sigma(\geq k), \tag{6}$$

and have calculated values for the cases $k=2,3$, and an estimate for the case $k=4$. They found that

$$\begin{aligned}\Sigma(\geq 2) &\simeq .00658;\\ \Sigma(\geq 3) &\simeq .00000148;\\ \Sigma(\geq 4) &\simeq 10^{-11}.\end{aligned} \tag{7}$$

These results are also in good agreement with those obtained analytically by Cardy [3], who using methods of conformal field theory has found the exact $N\to\infty$ behaviour of the probability $\Sigma_N(\geq k)$ on a rectangle instead of a square.

These new results suggest further elaboration. First, in the thermodynamic limit $N\to\infty$ spanning clusters become infinite clusters, that is their size diverges. If on one hand we expect finite clusters to have zero weight in the thermodynamic limit, on the other hand we expect that infinite clusters may have a finite weight. If it is so, and if we have such a cluster in a certain sample, we expect it to give a non zero contribution to the value of Y, which then will be non zero at least for that sample. Now from the recent results summarized above we know that the number of infinite clusters that may form at the critical point is sample dependent in the thermodynamic limit. This holds because the probability $\Sigma(\geq k)$ to find at least k infinite clusters in $\mathcal{L}_2$ is non zero for all k, as proved numerically by Shchur and Kosyakov at least for $k=2,3,4$. This means that if we could produce a large number of samples of the square lattice at $p=p_c$ we would observe samples displaying one infinite cluster, samples displaying two infinite clusters and so on, so that an histogram (number of samples with k infinite clusters)vs(k) would have a finite variance. Hence, if infinite clusters at the critical point have finite weight, we would have that the number of non zero weighted clusters is sample dependent in the thermodynamic limit. Consequently, also their weights would remain sample dependent. According to Derrida's statement on lack of self-averaging as sample dependence in the thermodynamic limit it is reasonable to conjecture that if infinite clusters at $p=p_c$ have finite

weight then critical two dimensional percolation on the square lattice lacks of self-averaging.

Proving or disproving the above conjecture might be not too difficult. Positive results would show that critical percolation and fragmentation in models of disordered systems have something deep in common. In our opinion percolation models could well be the starting point to find the foreseen universality underlying all those disordered models.

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